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BETTI NUMBERS OF PERFECT HOMOGENEOUS IDEALS

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In this paper we study the graded minimal free resolution of the ideal, I, of any arithmetically Cohen–Macaulay projective variety. First we determine the range of the shifts (twisting numbers) that can possibly occur in the resolution, in terms of the Hilbert function of I. Then we find conditions under which some of the twisting numbers do not occur. Finally, in some 'good' cases, all the Betti numbers are (recursively) computed, in terms of the Hilbert function of I or that of $\mathrm{Ext}_R^n(R/I,R)$, where R is a polynomial ring over a field and n is the height of I in R.

Introduction

This paper is extracted from the first part of the author's doctoral dissertation [7], and deals with the study of the graded minimal free resolution of any perfect homogeneous ideal, I, of a polynomial ring, R, over a field k; hence of any arithmetically Cohen–Macaulay projective variety.

The first point is to understand how to grade each $\mathrm{Tor}_i^R(I,k)$, and to carefully check that the several ways which naturally come to mind all induce the same grading on it.

Once this is done, it is possible to determine the number and the range of the shifts that can possibly appear in the graded minimal free resolution of I, in terms of the Hilbert function of I (Theorem 2.2).

The next natural step is to find under which conditions some of the shifts will not show up, and this is taken care of in Section 3.

In Section 4 we put all this at work by looking at particular cases in which all the Betti numbers can be determined, just by knowing the Hilbert function of I or, with the help of an extra condition, the Hilbert function of $\mathrm{Ext}_R^n(R/I,R)$ (where n is the height of I).

1. Graded modules and their resolutions

Let R be a polynomial ring in N indeterminates over a field k: $R=k[X_1,\dots,X_N]$. Thus R is a noetherian Cohen–Macaulay ring of Krull dimension N.

We consider the usual grading on R (determined by the total degree of polynomials). Then each graded piece R_i of R is a k-vector space of dimension $\binom{i+N-1}{N-1}$ (for all $i \geq 0$) and R_1 (finitely) generates R as a k-algebra.

The ideal $\mathfrak{m} = \bigoplus_{i>0} R_i$ is the unique *maximal homogeneous ideal* of R (that is, maximal among all homogeneous ideals), and, for most purposes, R can be treated as though it were an ordinary local ring with maximal ideal $\mathfrak{m}$ (see [6]).

Accordingly, if

$$(\dagger) \qquad \cdots \to F_i \xrightarrow{d_i} F_{i-1} \to \cdots \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{\varepsilon} M \to 0$$

is a free resolution of an R-module M, and we denote

$$N_i = \operatorname{Im} d_i = \begin{cases} \operatorname{Ker} d_{i-1} & \text{for } i \geq 2, \\ \operatorname{Ker} \varepsilon & \text{for } i = 1, \end{cases}$$

then, by analogy with the local case, we call ($\dagger$) a *minimal* free resolution if $N_i \subseteq \mathfrak{m} F_{i-1}$ for all $i \geq 1$, where $\mathfrak{m} = \bigoplus_{i>0} R_i$.

By *graded R-module* we mean an R-module M with a decomposition by k-vector spaces, $M = \bigoplus_{i \in \mathbb{Z}} M_i$ (with $M_i = 0$ for all $i < \alpha$ and some $\alpha \in \mathbb{Z}$), compatible with the R-module structure, which means $R_i M_j \subseteq M_{i+j}$ for all i and j.

A *graded homomorphism* is a homogeneous homomorphism of graded R-modules of degree 0 (i.e. preserving degrees).

A *graded* (minimal) free resolution of a graded R-module M, is a resolution like ($\dagger$), with all F_i's graded R-modules and all d_i's (and ε) graded homomorphisms.

If M is finitely generated, then every F_i has to be of the form $F_i = \bigoplus_{j=1}^{r} (R(-\gamma_{i,j}))^{\alpha_{i,j}}$, where the $\gamma_{i,j}$'s and the $\alpha_{i,j}$'s are, respectively, the degrees of the generators of $N_i = \operatorname{Im} d_i$ and the number of generators in each degree.

We call the numbers $\gamma_{i,j}$ the *twisting numbers* of M and each $\alpha_{i,j}$ the *multiplicity of $\gamma_{i,j}$ (at F_i)*.

The numbers

$$b_i = \textstyle\sum_j \alpha_{i,j} = \operatorname{rank}(F_i) = \dim_k \operatorname{Tor}_i^R(M, k)$$

(= minimal number of generators of N_i) are called the *Betti numbers* of M.

Recall that, if M, N are graded R-modules, then $M \otimes_k N$ is graded (over k) by

$$(M \otimes_k N)_t = \bigoplus_{p+q=t} (M_p \otimes_k N_q);$$

while $M \otimes_R N$ is graded (over k) by putting $M \otimes_R N = \operatorname{Coker} \phi$, where

$$\phi : M \otimes_k R \otimes_k N \to M \otimes_k N,$$

$$x \otimes a \otimes y \mapsto (ax \otimes y) - (x \otimes ay)$$

for $x \in M$, $a \in R$, $y \in N$. In other words,

$$(M \otimes_R N)_t = \frac{(M \otimes_k N)_t}{(\operatorname{Im} \phi)_t} = \frac{(M \otimes_k N)_t}{\langle \{ax \otimes y - x \otimes ay\}_t \rangle}$$

where $\langle\{ax\otimes y - x\otimes ay\}_t\rangle$ denotes the subspace generated by all differences $ax\otimes y - x\otimes ay$ ($a\in R$, $x\in M$, $y\in N$) of degree t (see [8, p. 186]).

The grading of the tensor product induces a grading on

$$\mathrm{Tor}_i^R(M,N) = H_i(\mathscr{F}\otimes_R N),$$

where $\mathscr{F}$ denotes a graded free resolution of M. It can be shown that such a grading is independent of the choice of $\mathscr{F}$. Moreover, it can be checked that, if $\mathscr{G}$ is a graded free resolution of N, then we have the same grading induced on $\mathrm{Tor}_i^R(M,N)$ viewed as $H_i(M\otimes_R \mathscr{G})$.

Finally, it is easy to see that the usual isomorphisms

$$\frac{N_i}{\mathfrak{m}N_i} \cong \frac{F_i}{\mathfrak{m}F_i} \cong F_i \otimes_R \frac{R}{\mathfrak{m}} = \mathrm{Tor}_i^R(M,\Bbbk) \quad \forall\, i\geq 1,$$

(the first of which follows from the minimality of the resolution, using the homogeneous version of Nakayama's Lemma) are all graded.

Thus, the several gradings that could be induced on $\mathrm{Tor}_i^R(M,\Bbbk)$, are in fact all the same.

Notation. For convenience of notation, we shall, from now on, use the symbol '$\equiv$' to denote a graded isomorphism.

Proposition 1.1. *Let M be a graded R-module.*

(a) *Suppose $M_t = (0)$ for all $t<d$ (for some d). Then*

$$(\mathrm{Tor}_i^R(M,\Bbbk))_t = (0) \quad \textit{for all } t<d+i.$$

(b) *Suppose $(M)_t = (0)$ for all $t>\sigma$ (for some σ). Then*

$$(\mathrm{Tor}_i^R(M,\Bbbk))_t = (0) \quad \textit{for all } t>\sigma+i.$$

Proof. In order to compute $\mathrm{Tor}_i^R(M,\Bbbk)$ we use the (graded) Koszul complex

$$0\to R(-N)\to\cdots\to R(-i)^{\binom{N}{i}}\to\cdots\to R(-1)^N\to R\to \Bbbk\to 0$$

to resolve $\Bbbk$ over R. When tensoring with M we get, for each i,

$$R(-i)^{\binom{N}{i}}\otimes_R M \equiv (R(-i)\otimes_R M)^{\binom{N}{i}} \equiv (M(-i))^{\binom{N}{i}}.$$

Now, in case (a) we have $(M(-i))_t = M_{t-i} = (0)$ for $t<d+i$; while in case (b) we have $(M(-i))_t = M_{t-i} = (0)$ for $t>\sigma+i$.

Thus, in both cases, the tensor product (hence, a fortiori, $\mathrm{Tor}_i^R(M,\Bbbk)$) vanishes in the indicated degrees. $\square$

Let $M=\bigoplus_{i\in\mathbb{Z}} M_i$ be a finitely generated graded R-module. The *Hilbert function* of M is the function $H(M,\cdot):\mathbb{Z}\to\mathbb{Z}$ defined by $H(M,i)=\dim_{\Bbbk} M_i$ $\forall\, i\in\mathbb{Z}$. It is well known that, for large i, the Hilbert function becomes a polynomial (the *Hilbert polynomial* of M) of degree one less than the Krull dimension of M.

It is easy to prove the additivity of the Hilbert function, which holds also for any (finite length) long exact sequence of finitely generated R-modules.

We also define the *first difference of the Hilbert function* of M as

$$\Delta H(M,i)=H(M,i)-H(M,i-1) \quad \forall i\in\mathbb{Z}.$$

Inductively, for every $r>1$, define the *rth difference of the Hilbert function* of M as

$$\Delta^r H(M,i)=\Delta^{r-1}H(M,i)-\Delta^{r-1}H(M,i-1) \quad \forall i\in\mathbb{Z}.$$

We let $\Delta^0 H(M,i)$ simply mean $H(M,i)$.

From the additivity of the Hilbert function we can derive the additivity of the rth difference of the Hilbert function, for any $r>0$.

For a (finitely generated) graded Cohen–Macaulay R-module M of positive Krull dimension, it is not hard to prove that

$$H(A,i-1)\le H(A,i) \quad \text{for all } i;$$

and, if $H(A,i-1)=H(A,i)$ for some i, then $H(A,i)=H(A,i+1)$.

In other words, in this case, the Hilbert function of M is either strictly increasing, or it strictly increases until it reaches a constant value.

Moreover, if $\bar{M}$ denotes the quotient of M by a regular sequence on M of degree 1 and length m, then $\Delta^m H(M,i)=H(\bar{M},i)$ $\forall i\in\mathbb{Z}$.

2. Twisting numbers of a perfect homogeneous ideal

In this section we deal with a perfect homogeneous ideal I of $R=k[X_1,\dots,X_N]$. Suppose I has height n; then the homological dimension of I is $n-1$ i.e. I has graded minimal free R-resolution:

$$(\dagger) \qquad 0\to F_{n-1}\xrightarrow{d_{n-1}}\cdots\to F_i\xrightarrow{d_i}\cdots\xrightarrow{d_1}F_0\to I\to 0.$$

We determine the number and the range of the twisting numbers of I, by showing that the degrees in which $\operatorname{Tor}_i^R(I,k)$ does not vanish (i.e. the degrees in which generators of N_i can be found) are related to the degrees in which generators of I can be found.

Proposition 2.1 (0-dimensional case). *Let J be a homogeneous ideal of $S=k[X_1,\dots,X_n]$ such that*

$$J=(J_d\oplus J_{d+1}\oplus\cdots\oplus J_{d+r-1})\oplus\mathfrak{m}^{d+r},$$

where $\mathfrak{m}=(X_1,\dots,X_n)$ (hence $\sqrt{J}=\mathfrak{m}$). Then $\operatorname{Tor}_i^S(J,k)$ vanishes in every degree different from $d+i,\dots,d+i+r$.

Proof. Since $J_t=(0)$ for all $t<d$, by part (a) of Proposition 1.1, we have $(\operatorname{Tor}_i^S(J,k))_t=(0)$ for every $t<d+i$. Now, put $B=S/J$. Then $\operatorname{Tor}_i^S(J,k)\cong$

$\mathrm{Tor}_{i+1}^S(B,k)$ $\forall i \geq 0$. But $B_t = (0)$ for all $t > d+r-1$, hence, from part (b) of Proposition 1.1, we get that $(\mathrm{Tor}_i^S(J,k))_t = (\mathrm{Tor}_{i+1}^S(B,k))_t = (0)$ $\forall t > d+r-1+i+1 = d+i+r$. □

Remark. The equality "$(\mathrm{Tor}_i^S(J,k))_t = (0)$ for $t > d+i+r$" could also be derived from [2, Proposition 1.1], without making use of part (b) of our Proposition 1.1.

Define

$$\alpha(I) = \min\{t \mid I_t \neq (0)\}.$$

Assume k is infinite (actually, it follows from [11, Lemma 1.1] that this restriction can be dropped). Then we may assume that $X_{n+1}, \ldots, X_N$ is a regular sequence modulo I - that is, a regular sequence on the R-module $A = R/I$. Put

$$B = \frac{A}{(X_{n+1}, \ldots, X_N)A} \equiv \frac{R}{(I, X_{n+1}, \ldots, X_N)}.$$

Then B has Krull dimension 0, hence its Hilbert polynomial is 0, and so B is eventually (0). Define

$$\sigma(I) = \min\{t \mid \Delta^{N-n} H(A,t) = 0\} = \min\{t \mid B_t = (0)\}.$$

It is clear that if $F_1, \ldots, F_h$ is a minimal set of generators of I, then

$$\min\{\deg F_i \mid i = 1, \ldots, h\} = \alpha(I);$$

and it can easily be proved that $\max\{\deg F_i \mid i = 1, \ldots, h\} \leq \sigma(I)$.

Theorem 2.2. *Let I be a height n perfect homogeneous ideal of $R = k[X_1, \ldots, X_N]$ and suppose $\alpha(I) = d$, $\sigma(I) = d+r$. Then* $\mathrm{Tor}_i^R(I,k)$ *vanishes in every degree different from*

$$d+i, d+i+1, \ldots, d+i+r.$$

Proof. Assume, as above, that $X_{n+1}, \ldots, X_N$ is a regular sequence modulo I, and put

$$J = \frac{(I, X_{n+1}, \ldots, X_N)}{(X_{n+1}, \ldots, X_N)}, \qquad S = \frac{R}{(X_{n+1}, \ldots, X_N)} \equiv k[X_1, \ldots, X_n].$$

Then we have that $\mathrm{Tor}_i^R(I,k) \equiv \mathrm{Tor}_i^S(J,k)$. Now, the assumptions on $\alpha(I)$ and $\sigma(I)$ yield

$$J = (J_d \oplus J_{d+1} \oplus \cdots \oplus J_{d+r-1}) \oplus \mathfrak{m}^{d+r}$$

where $\mathfrak{m} = (X_1, \ldots, X_n)$. Thus the result follows from Proposition 2.1. □

Examples. (a) As an application of Theorem 2.2, let us recover the resolution of the rational normal curve, $\mathscr{C}$, in $\mathbb{P}^n$ ($n \geq 3$). Its ideal, $\mathscr{P}$, is a (height $n-1$) perfect prime ideal of $k[X_0, \ldots, X_n]$, with $\alpha(\mathscr{P}) = 2$ and Hilbert function $H(\mathscr{P}, i) = \binom{i+n}{n} - (in+1)$ $\forall i \geq 2$. Thus its coordinate ring, $\mathscr{A}$, has Hilbert function

$$1 \quad n+1 \quad 2n+1 \quad \dots \quad in+1 \quad \dots,$$

hence $\Delta H(\mathscr{A}, \cdot)$ is given by

$$1 \quad n \quad n \quad \longrightarrow,$$

and so $\Delta^2 H(\mathscr{A}, \cdot)$ is

$$1 \quad n-1 \quad 0.$$

Therefore $\sigma(\mathscr{P}) = 2 = \alpha(\mathscr{P})$, i.e. $r=0$. Thus the resolution of $\mathscr{P}$ is *linear* (see [2]) and it is given by

$$0 \to R(-n)^{\alpha_{n-2}} \to \cdots \to R(-(2+i))^{\alpha_i} \to R(-(2+i-1))^{\alpha_{i-1}} \to \cdots$$
$$\cdots \to R(-3)^{\alpha_1} \to R(-2)^{\alpha_0} \to \mathscr{P} \to 0.$$

where all the α_i's can be computed.

(b) For another application of Theorem 2.2, let $I^{[s]}$ denote the ideal in $R = k[X_0, \dots, X_n]$ of s points in $\mathbb{P}^n$, where $\binom{d-1+n}{n} < s < \binom{d+n}{n}$, and let $A^{[s]}$ be their (homogeneous) coordinate ring. Suppose $H(A^{[s]}, d) = s$, i.e. the points impose independent conditions on the forms of degree d. Then $\alpha(I^{[s]}) = d$ and $\sigma(I^{[s]}) = d+1$ (see [5]). Thus, recalling also that $I^{[s]}$ is a perfect homogeneous ideal of height n, the resolution of the points looks like

$$0 \to R(-(d+n-1))^{\alpha_{n-1}} \oplus R(-(d+n))^{\beta_{n-1}} \to \cdots$$
$$\cdots \to R(-(d+i))^{\alpha_i} \oplus R(-(d+i+1))^{\beta_i} \to \cdots$$
$$\cdots \to R(-d)^{\alpha_0} \oplus R(-(d+1))^{\beta_0} \to I^{[s]} \to 0.$$

More generally, Theorem 2.2 states that, in a graded minimal free resolution of I over R, like (†), we have:

$$F_i = R(-(d+i))^{\alpha_{i,0}} \oplus R(-(d+i+1))^{\alpha_{i,1}} \oplus \cdots$$
$$\cdots \oplus R(-(d+i+r-1))^{\alpha_{i,r-1}} \oplus R(-(d+i+r))^{\alpha_{i,r}}$$

for each $i = 0, \dots, n-1$; with the $\alpha_{i,j}$'s not necessarily all different from 0. If we put (as in Section 1) $N_i = \operatorname{Im} d_i$, for every $i \geq 1$, and $N_0 = I$, then we have that $\alpha_{i,0} = H(N_i, d+i)$ and that N_i is generated at most in degrees $d+i, \dots, d+i+r = \sigma(I)+i$ (for all $i = 0, \dots, n-1$).

This last remark can be viewed as an extension to the whole resolution of the fact that $I = N_0$ is generated at most in degrees $d, \dots, d+r = \sigma(I)$.

We can dualize resolution (†), by applying the functor $(\cdot)^* = \operatorname{Hom}_R(\cdot, R)$, and obtain

$$(\dagger)^* \qquad 0 \to R \xrightarrow{\partial_0} F_0^* \xrightarrow{\partial_1} \cdots \to F_{i-1}^* \xrightarrow{\partial_i} F_i^* \to \cdots \to F_{n-1}^* \xrightarrow{\varepsilon} \operatorname{Ext}_R^n(A, R) \to 0,$$

which turns out to be a graded minimal free resolution of $\operatorname{Ext}_R^n(A, R)$, with

$$F_i^* = \bigoplus_{j=1}^{r} R(\gamma_{i,j})^{\alpha_{i,j}} \quad \forall i = 0, \dots, n-1.$$

Thus, Theorem 2.2 also tells us that in the dual resolution each F_i^* has the form

$$F_i^* = R(d+i+r)^{\alpha_{i,r}} \oplus R(d+i+r-1)^{\alpha_{i,r-1}} \oplus \cdots \oplus R(d+i+1)^{\alpha_{i,1}} \oplus R(d+i)^{\alpha_{i,0}}.$$

Hence, if we put $L_i = \operatorname{Im} \partial_i$, for each $i = 1, \dots, n-1$, and $L_n = \operatorname{Im} \varepsilon = \operatorname{Ext}_R^n(A, R)$, it follows that each L_i is generated at most in degrees $-(d+i-1+r), \dots, -(d+i-1)$, and $\alpha_{i-1,r} = H(L_i, -(d+i-1+r))$, for each $i = 1, \dots, n$.

Let us look more closely at $E = \operatorname{Ext}_R^n(A, R)$. First of all, because of the additivity of the Hilbert function, from (†)* we get that $E_t = (0)$ for $t < -d-n+1-r$, and $H(E, -(d+n-1+r) = \alpha_{n-1,r}$. We shall see in the next section that $\alpha_{n-1,r} \neq 0$, and so we have $\alpha(E) = -(d+n-1+r)$.

Now, using a result by Serre [10, p. IV-13, Proposition 5], it can be proved that there is a (non-graded) isomorphism

$$\operatorname{Ext}_R^n(A, R) \cong \frac{(H_1, \dots, H_n) : I}{(H_1, \dots, H_n)},$$

where $H_1, \dots, H_n$ is a regular sequence contained in I.

Furthermore, using [1, Theorem 3], it can be shown that

$$\Delta^{N-n} H\left(\frac{(H_1, \dots, H_n) : I}{(H_1, \dots, H_n)}, t\right) = \Delta^{N-n} H(A, l-t),$$

where $l = \sigma(H_1, \dots, H_n) - 1 = (\sum_{i=1}^n d_i) - n$, if $d_i = \deg H_i$ for all $i = 1, \dots, n$.

From all this, one can deduce that the graded version of the isomorphism above is

$$E\left(-\sum_{i=1}^{n} d_i\right) \cong \frac{(H_1, \dots, H_n) : I}{(H_1, \dots, H_n)},$$

and that $\Delta^{N-n} H(A, t) = \Delta^{N-n} H(E, -n-t)$.

3. Twisting numbers actually occurring

Now we want to find out under which conditions some of the $\alpha_{i,j}$'s can be 0, and how this can help in computing the Betti numbers.

First of all, since b_0 is the minimal number of generators of I (usually denoted by $\nu(I)$), if we assume that $\alpha(I) = d$, we must necessarily have $\alpha_{0,0} = H(I, d) \neq 0$.

On the other hand, assuming $X_{n+1}, \dots, X_N$ is a regular sequence modulo I - and putting

$$S = \frac{R}{(X_{n+1}, \dots, X_N)}, \quad J = \frac{(I, X_{n+1}, \dots, X_N)}{(X_{n+1}, \dots, X_N)}, \quad B = \frac{A}{(X_{n+1}, \dots, X_N)A}$$

- we have $\mathrm{Tor}^R_{n-1}(I,k) \cong \mathrm{Ann}_B(\mathfrak{m}/J)(-n)$, where $\mathfrak{m}=(X_1,\ldots,X_N)$ (see [7, Chapter I, (3.3)]). In particular,

$$b_{n-1}=\dim_k \mathrm{Tor}^R_{n-1}(I,k)=\dim_k \mathrm{Ann}_B(\mathfrak{m}/J),$$

which is known to be the Cohen–Macaulay type of A. Furthermore, as $\alpha(I)=d$ and $\sigma(I)=d+r$, we get

$$B=k\oplus S_1\oplus\cdots\oplus S_{d-1}\oplus\frac{S_d}{J_d}\oplus\cdots\oplus\frac{S_{d+r-1}}{J_{d+r-1}};$$

and hence

$$\left(\mathrm{Ann}_B\left(\frac{\mathfrak{m}}{J}\right)\right)_{d+i-r}=B_{d+r-1}=\frac{S_{d+r-1}}{J_{d+r-1}}\neq(0),$$

for $\sigma(I)=d+r$ implies $J_{d+r-1}\subsetneq S_{d+r-1}$. Thus,

$$\alpha_{n-1,r}=\dim_k(\mathrm{Tor}^R_{n-1}(I,k))_{\mathrm{d+n-1+r}}=\dim_k\left(\mathrm{Ann}_R\left(\frac{\mathfrak{m}}{J}\right)\right)_{d-1+r}\neq 0.$$

As for the vanishing of the $\alpha_{i,j}$'s, we first notice that it 'propagates' through the resolution, in the following sense:

Proposition 3.1. *Let I be a perfect homogeneous ideal of R of height n, with $\alpha(I)=d$ and $\sigma(I)=d+r$, and let $\alpha_{i,j}$ be the multiplicity of $d+i+j$ at F_i ($j=0,\ldots,r$; $i=0,\ldots,n-1$). Then*:

(a) *if $\alpha_{i,0}=\alpha_{i,1}=\cdots=\alpha_{i,u}=0$, for some i and for some u ($1\le i\le n-1$, $0\le u\le r-1$), then*

$$\alpha_{j,0}=\alpha_{j,1}=\cdots=\alpha_{j,u}=0,\quad \textit{for all } j\ge i;$$

(b) *if $\alpha_{i,r}=\alpha_{i,r-1}=\cdots=\alpha_{i,r-u}=0$, for some i and some u ($0\le i\le n-2$, $0\le u\le r-1$), then*

$$\alpha_{j,r}=\alpha_{j,r-1}=\cdots=\alpha_{j,r-u}=0,\quad \textit{for all } j\le i.$$

Proof. To prove (a), use induction on j. If $j=i$, there is nothing to prove. For $j>i$ by induction we have that

$$\alpha_{j-1,0}=\cdots=\alpha_{j-1,u}=0,$$

hence $(F_{j-1})_t=(0)$, for $t<d+j+u$. Therefore $(\mathfrak{m}F_{j-1})_t=(0)$, for $t<d+j+u+1$, which means that $\alpha_{j,0}=\cdots=\alpha_{j,u}=0$.

Similarly, (b) follows from the minimality of the dual resolution. □

Now, for any $i=0,\ldots,n-1$ and every $u=1,\ldots,r$, let $W_u(N_i)$ denote the vector subspace of $(N_i)_{d+i+u}$ generated by $(N_i)_{d+i+u-1}$ under multiplication by $X_1,\ldots,X_N$, i.e.,

$$W_u(N_i)=X_1(N_i)_{d+i+u-1}+\cdots+X_N(N_i)_{d+i+u-1}\subseteq(N_i)_{d+i+u}.$$

It is clear that for each $i\ge 0$ and $u=1,\ldots,r$,

$$\dim_k W_u(N_i) = H(N_i, d+i+u) - \alpha_{i,u}.$$

Similarly, consider the dual resolution and, for any $i=1,\dots,n$ and every $u=1,\dots,r$, define $W_u(L_i)$ as the k-vector subspace of $(L_i)_{-(d+i-1+r-u)}$ generated by $(L_i)_{-(d+i+r-u)}$ under multiplication by $X_1,\dots,X_N$, i.e.,

$$W_u(L_i) = X_1(L_i)_{-(d+i+r-u)} + \cdots + X_N(L_i)_{-(d+i+r-u)} \subseteq (L_i)_{-(d+i-1+r-u)}.$$

Again, it is clear that

$$\dim_k W_u(L_i) = H(L_i, -(d+i-1+r-u)) - \alpha_{i-1,r-u}.$$

Theorem 3.2. *Let I be a perfect homogeneous ideal of R of height n, with $\alpha(I)=d$, $\sigma(I)=d+r$; and let $\alpha_{i,j}$ be the multiplicity of $d+i+j$ at F_i ($j=0,\dots,r$; $i=0,\dots,n-1$). Then*:

(a) *for any $i=1,\dots,n-1$ and any $u=0,\dots,r-1$,*

$$\alpha_{k,0} = \alpha_{k,1} = \cdots = \alpha_{k,u} = 0 \quad \forall k=i,\dots,n-1,$$

if and only if

$$\sum_{j=0}^{u+1} \binom{u+1-j+N-1}{N-1} \alpha_{i-1,j} = H(N_{i-1}, d+i+u)$$

or, equivalently, if and only if

$$\dim_k W_{u+1}(N_{i-1}) = \sum_{j=0}^{u} \binom{u+1-j+N-1}{N-1} \alpha_{i-1,j};$$

(b) *for any $i=1,\dots,n-1$ and any $u=0,\dots,r-1$,*

$$\alpha_{k,r} = \alpha_{k,r-1} = \cdots = \alpha_{k,r-u} = 0 \quad \forall k=0,\dots,i-1,$$

if and only if

$$\sum_{j=0}^{u+1} \binom{u+1-j+N-1}{N-1} \alpha_{i,r-j} = H(L_{i+1}, -(d+i-1+r-u))$$

or, equivalently, if and only if

$$\dim_k W_{u+1}(L_{i+1}) = \sum_{j=0}^{u} \binom{u+1-j+N-1}{N-1} \alpha_{i,r-j}.$$

Before proving the theorem above we need a lemma.

Lemma 3.3. *Let $M = M_a \oplus M_{a+1} \oplus \cdots \oplus M_{a+r} \oplus \cdots$ be a graded R-module (finitely) generated in degrees $a, a+1, \dots, a+r$. For every $u=1,\dots,r$, put $W_u = X_1 M_{a+u-1} + \cdots + X_N M_{a+u-1} \subseteq M_{a+u}$, and $\mu_u = H(M, a+u) - \dim_k W_u$. Also put $\mu_0 = H(M,a)$. Then, for any $u=1,\dots,r$,*

$$\dim_k W_u = \sum_{j=0}^{u-1} \binom{u-j+N-1}{N-1} \mu_j$$

if and only if

$$\dim_k W_v = \sum_{j=0}^{v-1} \binom{i-j+N-1}{N-1} \mu_j \quad \forall v = 1, \ldots, u.$$

Proof. Consider the graded short exact sequence

$$0 \to N_1 \to \bigoplus_{j=1}^{r} R(-a-j)^{\mu_j} \to M \to 0,$$

which is a graded minimal presentation of M. Thus $(N_1)_t = (0)$ $\forall t < a+1$, from part (a) of Proposition 1.1.

For simplicity of notation, put $N_1 = N$.

From the exact sequence above we obtain, for each degree t, a short exact sequence of k-vector spaces

$$0 \to N_t \to \left(\bigoplus_{j=1}^{r} R(-a-j)^{\mu_j} \right)_t \to M_t \to 0.$$

In particular, if we fix any $u = 1, \ldots, r$, then, $\forall v = 1, \ldots, u$, we have

$$\begin{aligned} \dim_k M_{a+v} &= \sum_{j=1}^{r} \binom{v-j+N-1}{N-1} \mu_j - \dim_k N_{a+v} \\ &= \sum_{j=1}^{v} \binom{v-j+N-1}{N-1} \mu_j - \dim_k N_{a+v}. \end{aligned}$$

On the other hand, $\dim_k M_{a+v} = H(M, a+i) = \dim_k W_v + \mu_v$, and so

$$\begin{aligned} \dim_k N_{a+v} &= \dim_k W_v + \mu_v - \sum_{j=1}^{v} \binom{v-j+N-1}{N-1} \mu_j \\ &= \dim_k W_v - \sum_{j=1}^{v-1} \binom{v-j+N-1}{N-1} \mu_j. \end{aligned}$$

Thus, the statement we want to prove is equivalent to

$$N_{a+u} = (0) \quad \Leftrightarrow \quad N_{a+v} = (0) \quad \forall v = 1, \ldots, u.$$

But this is clear, since N is a submodule of a free module and $N_t = (0)$ for all $t < a+1$. $\square$

Proof of Theorem 3.2. First of all, in view of Proposition 3.1(a), it is sufficient to prove (a) for $k = i$. Consider the graded short exact sequence

$$0 \to N_i \to F_{i-1} \to N_{i-1} \to 0$$

where

$$F_{i-1} = R(-(d+i-1))^{\alpha_{i-1,0}} \oplus \cdots \oplus R(-(d+i-1+u))^{\alpha_{i-1,u}} \oplus$$

$$\oplus R(-(d+i+u))^{\alpha_{i-1,u+1}} \oplus R(-(d+i+u+1))^{\alpha_{i-1,u+2}} \oplus \cdots$$
$$\cdots \oplus R(-(d+i-1+r))^{\alpha_{i-1,r}}.$$

If $\alpha_{i,0} = \cdots = \alpha_{i,u-1} = 0$, then, from the additivity of the Hilbert function, we get that

$$\begin{aligned} 0 = \alpha_{i,u} &= H(N_i, d+i+u) \\ &= \binom{u+1+N-1}{N-1}\alpha_{i-1,0} + \binom{u+N-1}{N-1}\alpha_{i-1,i} + \cdots \\ &\quad \cdots + N\alpha_{i-1,u} + \alpha_{i-1,u+1} - H(N_{i-1}, d+i+u) \\ &= \left(\sum_{j=0}^{u+1}\binom{u+1-j+N-1}{N-1}\alpha_{i-1,j}\right) - H(N_{i-1}, d+i+u) \\ &= \left(\sum_{j=0}^{u}\binom{u+1-j+N-1}{N-1}\alpha_{i-1,j}\right) - \dim_k W_{u+1}(N_{i-1}). \end{aligned}$$

Conversely, if

$$\dim_k W_{u+1}(N_{i-1}) = \sum_{j=0}^{u}\binom{u+1-j+N-1}{N-1}\alpha_{i-1,j},$$

then, from Lemma 3.3,

$$\dim_k W_{t+1}(N_{i-1}) = \sum_{j=0}^{t}\binom{t+1-j+N-1}{N-1}\alpha_{i-1,j}$$

for all $t = 0, \ldots, u$. Hence, by recursively applying the additivity of the Hilbert function as above, we obtain $\alpha_{i,t} = 0 = H(N_i, d+i+t)$, for all $t = 0, \ldots, u$.

The proof of (b) is perfectly similar: use Proposition 3.1(b), then consider the graded short exact sequence

$$0 \to L_i \to F_i^* \to L_{i+1} \to 0,$$

with

$$F_i^* = R(d+i+r)^{\alpha_{i,r}} \oplus \cdots \oplus R(d+i+r-u)^{\alpha_{i,r-u}}$$
$$\oplus R(d+i+r-u-1)^{\alpha_{i,r-u-1}} \oplus \cdots \oplus R(d+i)^{\alpha_{i,0}},$$

and finally apply Lemma 3.3 to $W_{u+1}(L_{i+1})$. $\square$

4. Special cases

Now we specialize Theorem 3.2 to particular situations in which a 'good' piece of information about I allows us to determine all the Betti numbers.

Proposition 4.1. *Let I be a perfect homogeneous ideal of R of height n, with $\alpha(I) = d$ and $\sigma(I) = d + r$, and let $\alpha_{i,j}$ be the multiplicity of $d+i+j$ at F_i ($j = 0, \ldots, r$;*

$i=0,\dots,n-1$). *Assume* $\alpha_{0,0}=1$, *and, for some* u $(0\le u\le r-1)$, $\alpha_{0,j}=0$ $\forall j=1,\dots,u$. *Then*

$$\alpha_{i,0}=\alpha_{i,1}=\cdots=\alpha_{i,u}=0 \quad \forall i=1,\dots,n-1.$$

Proof. Let G be a basis of $I_d=(N_0)_d$. Then, $\forall t=1,\dots,u+1$, $W_t(N_0)=R_tG$. On the other hand, multiplication by G gives an injective map $R_t\to R_{t+d}$, thus $R_tG\cong R_t$, therefore

$$\dim_k W_t(N_0)=\binom{t+N-1}{N-1} \quad \forall t=1,\dots,u+1.$$

Moreover, since $\alpha_{0,0}=1$ and $\alpha_{0,j}=0$ $\forall j=1,\dots,u$, we have that

$$\sum_{j=0}^{t-1}\binom{t-j+N-1}{N-1}\alpha_{0,j}=\binom{t+N-1}{N-1} \quad \forall t=1,\dots,u+1;$$

and so we get

$$\dim_k W_t(N_0)=\sum_{j=0}^{t-1}\binom{t-j+N-1}{N-1} \quad \forall t=1,\dots,u+1.$$

Now the conclusion follows from part (a) of Theorem 3.2. $\square$

If Proposition 4.1 holds for $u=r-1$ (which means, when I is generated by one form of lowest degree and all other generators are of top degree), then the resolution of I is linear except at F_0, with $F_0=R(-d)\oplus R(-(d+r))^{\alpha_{0,r}}$ and $F_i=R(-(d+i+r))^{\alpha_{i,r}}$ for every $i=1,\dots,n-1$. We call such a resolution *right almost linear*. In this case the Betti numbers of I can be computed, since $\alpha_{0,r}=H(I,d+r)-\dim_k W_r(I)=H(I,d+r)-\binom{r+N-1}{N-1}$ and $b_0=1+\alpha_{0,r}$, while $b_i=\alpha_{i,r}=H(N_i,d+i+r)$ $\forall i=1,\dots,n-1$. The latter can be recovered from the resolution by applying the additivity of the Hilbert function to the graded long exact sequence:

$$0\to N_i\to R(-(d+i-1+r))^{\alpha_{i-1,r}}\to\cdots\to R(-(d+1+r))^{\alpha_{1,r}}$$
$$\to R(-d)\oplus R(-(d+r))^{\alpha_{0,r}}\to I\to 0.$$

Furthermore, Proposition 4.1 can be generalized, by means of the following lemma, which is known.

Lemma 4.2. *Let* $H_1,\dots,H_\lambda$ *be a regular sequence in* R *of forms of degree* d *and let* $K=(H_1,\dots,H_\lambda)$. *Then, for every* $u<d$,

$$\dim_k K_{d+u}=\lambda\binom{u+N-1}{N-1}.$$

Proposition 4.3. *If* I_d *contains a regular sequence of length* λ $(\lambda\le n)$, *then Proposition* 4.1 *holds also for* $\alpha_{0,0}=\lambda$ *and any* $u<d-1$.

Proof. Let $H_1,\dots,H_\lambda \in I_d$ be a regular sequence and put $K=(H_1,\dots,H_\lambda)$. Since $\alpha_{0,j}=0$ $\forall j=1,\dots,u$, we get:

$$W_t(N_0)=K_{d+t} \quad \forall t=1,\dots,u+1.$$

Thus, from the previous lemma, we get

$$\dim_k W_t(N_0)=\lambda\binom{t+N-1}{N-1} \quad \forall t=1,\dots,u+1;$$

and (of course) we have:

$$\sum_{j=0}^{t-1}\binom{t-j+N-1}{N-1}\alpha_{0,j}=\lambda\binom{t+N-1}{N-1} \quad \forall t=,\dots,u+1,$$

as $\alpha_{0,j}=0$ $\forall j=1,\dots,u$ and $\alpha_{0,0}=\lambda$. The conclusion now follows after applying part (a) of Theorem 3.2, as in Proposition 4.1. □

Note that the restriction on u is forced by Lemma 4.2, while the restriction on λ is a direct consequence of requiring the existence of a regular sequence of length λ in I_d.

If Proposition 4.3 holds for $u=r-1$ (hence, necessarily, $r\le d-1$, i.e., $\sigma(I)\le 2d-1$), then, again, we have a right almost linear resolution (with $F_0=R(-d)^\lambda\oplus R(-d-r)^{\alpha_{0,r}}$ and $F_i=R(-d-i-r)^{\alpha_{i,r}}$ $\forall i=1,\dots,n-1$) and, again, all the Betti numbers of I can be recovered.

A dual version of Proposition 4.1 is not as straightforward: unlike $N_0=I$ and each N_i and L_i $(i=0,\dots,n-1)$, $L_n=E=\operatorname{Ext}^n_R(A,R)$ need *not* be a submodule of a free module; thus $\alpha_{n-i,r}=1$ and $\alpha_{n-1,r-1}=\cdots=\alpha_{n-1,r-u}=0$ are not enough to guarantee that $\dim_k W_{u+1}(E)=\binom{u+1+N-1}{N-1}$. However,

Proposition 4.4. *Let I be a perfect homogeneous ideal of R of height n, with $\alpha(I)=d$ and $\sigma(I)=d+r$. Let $\alpha_{i,j}$ be the multiplicity of $d+i+j$ at F_i $(j=0,\dots,r$; $i=0,\dots,n-1)$. Assume $\alpha_{n-1,r}=1$ and $\alpha_{n-1,r-1}=\cdots=\alpha_{n-1,r-u}=0$, for some u, $0\le u\le r-1$. Assume also that $(\operatorname{Ann}_R(x))_t=(0)$ for all $t\le u+1$ and all nonzero $x\in E_{-(d+n-1+r)}$. Then*

$$\alpha_{i,r}=\alpha_{i,r-1}=\cdots=\alpha_{i,r-u}=0 \quad \forall i=0,\dots,n-1.$$

Proof. The condition on the annihilator makes sure that

$$\dim_k W_{t+1}(E)=\binom{t+1+N-1}{N-1}=\sum_{j=0}^{t}\binom{t+1+N-1}{N-1}\alpha_{n-1,r-t},$$

for all $t\le u+1$. Then the result follows by applying part (b) of Theorem 3.2. □

Note that necessarily $u<d-1$. In fact,

$$E\cong\frac{(H_1,\dots,H_n):I}{(H_1,\dots,H_n)}$$

(where $H_1, \dots, H_n$ is a regular sequence contained in I), thus,

$$\mathrm{Ann}_R(E) = \mathrm{Ann}_R\left(\frac{(H_1,\dots,H_n):I}{(H_1,\dots,H_n)}\right) = (H_1,\dots,H_n):((H_1,\dots,H_n):I) \supseteq I.$$

Hence $\mathrm{Ann}_R(x) \supseteq \mathrm{Ann}_R(E) \supseteq I$ $\forall x \in E$; and so $(\mathrm{Ann}_R(x))_t \neq 0$ for $t \geq d$.

If Proposition 4.4 holds for $u = r-1$ (hence, necessarily, $r \leq d-1$), the resolution of I is *almost linear* [2], $F_i = R(-(d+i))^{\alpha_{i,0}}$ $\forall i = 0, \dots, n-2$, and $F_{n-1} = R(-(d+n-1))^{\alpha_{n-1,0}} \oplus R(-(d+n-1+r))$. In this case, the Betti numbers b_i ($\forall i = 0, \dots, n-2$) can be recovered from the long exact sequence

$$0 \to N_i \to R(-(d+i-1))^{\alpha_{i-1,0}} \to \cdots \to R(-d)^{\alpha_{0,0}} \to I \to 0;$$

while $b_{n-1} = \alpha_{n-1,0} + 1$, where $\alpha_{n-1,0} = H(E, -(d+n-1)) - \binom{r+N-1}{N-1}$.

An interesting application of Proposition 4.4 is when $A = R/I$ is Gorenstein. In this case we necessarily have $\alpha_{n-1,r} = 1$ and $\alpha_{n-1,r-1} = \cdots = \alpha_{n-1,1} = \alpha_{n-1,0} = 0$.

We also have a (non graded) isomorphism $E \cong A$, so we get

$$b_i = b_{n-2-i} \quad \forall i = 0, \dots, n-2 \quad (b_{n-1} = 1),$$

by comparing a resolution of A with its dual resolution (see [11]).

From the isomorphism above we also get that every non-zero $x \in E_{-(d+n-1+r)}$ is a generator of E, thus

$$\mathrm{Ann}_R(x) = \mathrm{Ann}_R(E) = \mathrm{Ann}_R(A) = I.$$

Hence, $\forall x \in E_{-(d+n-1+r)}$, $x \neq 0$, we have $(\mathrm{Ann}_R(x))_t = (0)$ for $t < d$. This means we can apply Proposition 4.4 for $u = d-2$, to get

$$\alpha_{i,r} = \alpha_{i,r-1} = \cdots = \alpha_{i,r-d+2} = 0 \quad \forall i = 0, \dots, n-2.$$

It is not hard to see that the graded version of the isomorphism above is $E \equiv A(d+n-1+r)$. Thus, twisting the resolution of A by $d+n-1+r$ and then comparing it with the dual resolution, we obtain

$$\alpha_{i,j} = \alpha_{n-2-i,r-d+1-j}$$

for all $j = 0, \dots, r-d+1$ and all $i = 0, \dots, n-2$.

Note that, when A is Gorenstein, the condition on the annihilator stated in Proposition 4.4 is satisfied for $u = r-1$ if and only if A is *extremal Gorenstein* (i.e., Gorenstein with $r = d-1$). In this case the resolution of I is almost linear and pure; in fact it is linear except for a jump in degree at the last step:

$$F_i = R(-(d+i))^{\alpha_{i,0}} \quad \forall i = 0, \dots, n-2$$

and

$$F_{n-1} = R(-(d+n-1+r)).$$

Apart from the Gorenstein case, it is not easy, in general, to determine in which cases the condition on the annihilator stated in Proposition 4.4 is satisfied. If we think of E as

$$\frac{(H_1,\dots,H_n):I}{(H_1,\dots,H_n)}\left(\sum_{i=1}^{n} d_i\right);$$

where $H_1,\dots,H_n$ is a regular sequence contained in I and where $d_i=\deg H_i$ (for $i=1,\dots,n$), then the condition

$$\text{“}(\mathrm{Ann}_R(x))_t=(0)\quad \forall t\le u+1 \text{ and } \forall x\in E_{-(d+n-1+r)},\ x\ne 0\text{”}$$

becomes:

$$\text{“}\forall\ G\in((H_1,\dots,H_n):I)_{\sum d_i-(d+n-1+r)},\ G\notin(H_1,\dots,H_n),$$

$$((H_1,\dots,H_n):G)_t=(0)\quad \forall\ t\le u+1\text{”}.$$

In geometrical terms: the zero set of I is contained in the complete intersection determined by $H_1,\dots,H_n$ and we require that no hypersurface of degree less than or equal to $u+1$ vanish on that part of the complement of the zero set of G which is contained in the complete intersection.

Now we apply the ‘extremal’ cases of Theorem 3.2 to derive from I or E some information about the whole resolution.

Proposition 4.5. *Let I be a perfect homogeneous ideal of R of height n, with $\alpha(I)=d$ and $\sigma(I)=d+r$, and let $\alpha_{i,j}$ be the multiplicity of $d+i+j$ at F_i ($j=0,\dots,r$; $i=0,\dots,n-1$). Then*:

(a) *the resolution of I is right almost linear if and only if*

$$\sum_{j=0}^{r}\binom{r-j+N-1}{N-1}\alpha_{0,j}=H(I,d+r)$$

or (equivalently) if and only if

$$\sum_{j=0}^{r}\binom{r-j+n-1}{n-1}\alpha_{0,j}=\binom{d+r+n-1}{n-1};$$

(b) *the resolution of I is almost linear if and only if*

$$\sum_{j=0}^{r}\binom{r-j+N-1}{N-1}\alpha_{n-1,r-j}=H(E,-(d+n-1))$$

or, (equivalently) if and only if

$$\sum_{j=0}^{r}\binom{j+n-1}{n-1}\alpha_{n-1,j}=\binom{d-1+n-1}{n-1}.$$

Proof. The first parts of both (a) and (b) are simply respectively (a) and (b) of Theorem 3.2 for $u=r-1$ and $i=1$ (resp. $i=n-1$). The second parts are obtained by reducing modulo $X_{n+1},\dots,X_N$, which is a regular sequence modulo I. □

Remark. A condition equivalent to (b) is also contained in [2, Theorem A.1].

Notice that, since $\alpha_{0,0} \neq 0$ and $\alpha_{n-1,r} \neq 0$, the only way we could get a linear resolution is if the twisting numbers of I are $d, d+1, \dots, d+n-1$, i.e., the case $r=0$ (like the rational normal curve), which means that $A=R/I$ is *extremal Cohen-Macaulay* (i.e., Cohen-Macaulay with $\sigma(I)=\alpha(I)=d$ - see [9]).

In other words, as observed also in [2], I has a linear resolution if and only if

$$J=\frac{(I, X_{n-1}, \dots, X_N)}{(X_{n+1}, \dots, X_N)}=\mathfrak{m}^d$$

where $\mathfrak{m}=(X_1, \dots, X_N)$. In this case J is a determinantal ideal and can be dealt with by using the 'Eagon–Northcott technique' (see [3] and [4]).

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References

[1] E. Davis, A.V. Geramita and F. Orecchia, Gorenstein algebras and the Cayley–Bacharach theorem, Proc. Amer. Math. Soc. 93 (1985) 593–597.
[2] D. Eisenbud and S. Goto, Linear free resolutions and minimal multiplicity; J. Algebra 88 (1984) 89–133.
[3] J.A. Eagon and D.G. Northcott, Ideals defined by matrices and a certain complex associated with them, Proc. Roy. Soc. London Ser. A 269 (1962) 188–204.
[4] J.A. Eagon and D.G. Northcott, A note on the Hilbert functions of certain ideals which are defined by matrices, Mathematica 9 (1962) 118–126.
[5] A.V. Geramita and P. Maroscia, The ideal of forms vanishing at a finite set of points in $\mathbb{P}^n$, J. Algebra 90 (1984) 528–555.
[6] A. Lorenzini, Graded rings, Queen's Papers in Pure and Applied Mathematics 61 (1982) (The Curves Seminar at Queen's - Vol. II) Section D.
[7] A. Lorenzini, On the Betti numbers of points in projective space, Thesis, Queen's University, Kingston, Ontario, 1987.
[8] S. MacLane, Homology (Springer, Berlin, 1975).
[9] P. Schenzel, Über die freien Auflösungen extremaler Cohen-Macaulay Ringe, J. Algebra 64 (1980) 93–101.
[10] J.P. Serre, Algèbre local - Multiplicités, Lect. Notes in Mathematics 11 (Springer, Berlin, 1965).
[11] R.P. Stanley, Hilbert functions of graded algebras; Adv. Math. 28 (1978) 57–83.